HOW TO GROW YOUR OWN TREES FROM GIVEN CUT-SET OR TIE-SET MATRICES*

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Abstract

The method recognizes that construction of a tree (and hence the pertinent graph) from a given matrix can be done by inspection once the pattern of its growth has been established. To this end it is only necessary that we have a mechanism, applicable to a given cut-set matrix, which sorts out those rows that correspond to the outermost twigs or tips of the tree, for we can then form an abbridged cut-set matrix corresponding to what is left of the total graph after the tree tips with their uniquely attached links are pruned away. This remainder again has tips which can be found and eliminated in the same way. Continuation thus reveals the desired growth pattern.

Since the method cannot fail to yield a graph if its existence is compatible with the structure of the given matrix, it may be regarded as a constructive test for fulfillment of necessary and sufficient conditions.

1. Introductory Remarks

In this paper a cut-set matrix is not defined in the restricted (and less useful) form in which mathematicians define it when discussing purely topological questions, but rather as engineers define it in connection with network theory. Thus this matrix is not necessarily restricted to the form in which its elements are plus or minus one or zeros; they may be any real numbers, as they are when the voltage variables defined by this matrix are arbitrary linear combinations of the branch voltages. Similarly a tie-set (or loop-set) matrix has any real elements when the loop-current variables are defined as arbitrary linear combinations of the branch currents.

If the number of branches in the network is $\underline{b}$, the number of independent nodes $\underline{n}$ (one less than the total nodes n_t) and the number of independent loops $\ell = b - n$, then the cut-set matrix α has $\underline{n}$ rows and $\underline{b}$ columns, while the tie-set matrix β has $\underline{\ell}$ rows and $\underline{b}$ columns. Given such a matrix with real coefficients, our problem is to determine whether an associated network exists and to find the graph of that network together with the appropriate relations defining the pertinent voltage or current variables.

As is well known[1], the procedure for choosing voltage or current variables in the approach to a network analysis problem may take one of three basically different forms: (a) One may select a tree and choose the tree-branch voltages or the link currents as variables; (b) One may make a forthright choice of closed paths for loop currents or of node pairs for node-pair voltages; (c) One may initiate the proceedings algebraically by writing defining equations for currents or voltages expressing these as any desired linear combinations of the respective branch quantities.

Of these, the procedure under (a) seems most fundamental, and I would therefore like to refer to a particular form of the resulting α or β matrix in this case as its "fundamental" or "canonic" form. Any one of the forms arrived at under (b) or (c) are, of course, obtainable from a canonic form through making an appropriate combination of its rows, corresponding to premultiplication by a real nonsingular matrix characterizing the transformation from the variables as defined under (a) to those pertinent to (b) or (c). This fact immediately shows how we can obtain the canonic form of α or β from any other given one.

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For example, if we have the Kirchhoff current-law equations expressed by

$$\alpha j = i_s \tag{1}$$

in which j is a column matrix with the branch currents $j_1 \ldots j_b$ as elements and i_s is a column matrix involving the source currents $i_{s1} \ldots i_{sn}$ consistent with the voltage variables $e_1 \ldots e_n$, then (as is well known) these are related to the branch voltages $v_1 \ldots v_b$ in a column matrix v by

$$\alpha_t e = v \tag{2}$$

in which α_t denotes the transpose of α. Since a transformation to new variables with the column matrix e' may be written

$$e = \tau e' \tag{3}$$

where τ is a real nonsingular matrix of order n, we have straightforwardly

$$\alpha_t \times \tau \times e' = \bar{\alpha}_t e' = v. \tag{4}$$

The new α matrix is thus related to the old one by

$$\alpha_t \times \tau = \bar{\alpha}_t \text{ or } \bar{\alpha} = \tau_t \alpha \tag{5}$$

and Eq. 1 becomes

$$\tau_t \times \alpha \times j = \bar{\alpha} j = \tau_t i_s = i_s' \tag{6}$$

in which the elements $i_{s1}' \ldots i_{sn}'$ in the column matrix i_s' are new source currents appropriate to the new voltage variables in e'.

The result of chief interest here is expressed by Eq. 5 showing that the new α matrix $\bar{\alpha}$ (as stated above) is related to the old one through premultiplication by a transformation matrix τ_t. As might have been expected at the outset, the rows in $\bar{\alpha}$ (the new cut sets appropriate to the new node-pair voltages) are linear combinations of the rows (original cut sets) in α.

Thus, no matter what the form of the given α matrix may be, or by which fundamental process it may have originated, we can always assume that its canonic form is obtainable through premultiplication by a suitably chosen transformation matrix. The latter can, therefore, be constructed in any given situation in the following way: Since the cut sets defined by the given α matrix must be independent, it must be possible to find n independent columns. The inverse of the n x n matrix defined by these columns yields the desired transformation, for the given α matrix premultiplied by it results in an α matrix containing n columns like those in a unit matrix of order n, and this situation characterizes a cut-set matrix in canonic form. We prefer to arrange the columns in the canonic form so that a partitioning of them into groups of $\underline{\ell}$ and $\underline{n}$ may be written

$$\alpha = \left[\alpha_{n\ell} \;\vdots\; u_n\right] \tag{7}$$

in which $\alpha_{n\ell}$ is a submatrix having $\underline{n}$ rows and $\underline{\ell}$ columns, and u_n is a unit matrix of order $\underline{n}$. This arrangement assumes a branch numbering sequence that begins with the links pertinent to a particular choice of tree and ends with the successive numbering of tree branches in an order corresponding to the numbering of the voltage variables $e_1 \ldots e_n$.

In an exactly analogous fashion (dual to the foregoing) a given tie-set matrix can be converted to a canonic form which we shall write

$$\beta = \left[u_\ell \;\vdots\; \beta_{\ell n} \right] \tag{8}$$

where u_ℓ is a unit matrix of order $\underline{\ell}$ and $\beta_{\ell n}$ is a submatrix having $\underline{\ell}$ rows and $\underline{n}$ columns.

It is furthermore well known[2] that if α in 7 and β in 8 pertain to the same network and the same choice of tree then β is the negative of the transpose of $\alpha_{n\ell}$ and vice versa; that is

$$\beta_{\ell n} = -(\alpha_{n\ell})_t \text{ or } \alpha_{n\ell} = -(\beta_{\ell n})_t \tag{9}$$

For this reason it makes little difference whether the given matrix defines tie sets or cut sets in the desired graph since we can readily construct both α and β in canonic form from either given data.

If the given α or β matrix is assumed to have arisen through fundamental process (c) applied to some network whose graph we are to regain, then (as mentioned before)its elements may not be just plus ones, minus ones, or zeros, but any real numbers. A necessary condition that an associated graph exists is evidently that the resulting submatrix $\alpha_{n\ell}$ or $\beta_{\ell n}$ in the canonic form 7 or 8 gotten by the method described above shall consist exclusively of elements that are plus ones, minus ones, or zeros. The reason is obvious from the foregoing discussion; and it is also obvious that the canonic form of an α or β matrix is not unique but that (except for variants due to renumbering of branches) there are as many of these as there are distinct geometrical tree configurations associated with a given graph.*

2. The basic idea in the proposed method.

The underlying thought in the method of graph construction proposed here is based upon two considerations, namely: (a) that the essential part of the graph is the tree; once we have its configuration, the links are readily inserted by inspection of the rows in $\alpha_{n\ell}$; and (b) the tree can also be constructed by inspection of these rows if only we know the proper order in which to consider the various branches; that is to say, we must discover its pattern of growth from a seedling to its present state.

The sketches in Fig. 1 indicate what we mean by the latter statement. In part (a) we have a given tree; in this example it is three years old. The growth during the third year we call the "tips" of the tree. They are indicated by the short bars drawn across them. Part (b) of this series of sketches shows the remnant of the original tree after removal of the first set of tips (the third year's growth). Again we have tips representing the second year's growth. In part (c) these are removed showing the tree as it has developed from a seedling after the first year. Finally in part (d) we have the central node or the seedling itself.

Thus by removing successively the first set of tips from a given tree, then the second set of tips, and so forth, we can discover its pattern of growth. If in a given matrix $\alpha_{n\ell}$ we can arrange the rows in groups representing the first set of tips, then the second set, and so forth down to rows representing branches which sprout from the seedling during the first year's growth, then construction of the graph by inspection presents no problem and proceeds in an orderly straightforward manner. A simple example will illustrate what is meant here.

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* Although not important for the present discussion, this number should not be confused with the total number of trees as ordinarily considered. If an (n + 1) - node graph contains the maximum of n(n + 1)/2 branches, then the number of distinct geometrical tree configurations equals the number of different tree patterns constructible from n match sticks. For n = 2, 3, 4, 5, 6, 7, 8, ... this number is respectively 1, 2, 3, 6, 11, 23, 47, ..., which is not as astronomical as the total number of trees.

The schedule in the upper left-hand corner of Fig. 2 is a matrix $\alpha_{n\ell}$. The rows refer to correspondingly numbered tree branches and the columns to links. The rows are moreover arranged in such a way that the natural growth of the tree begins at the bottom with branch 12. This branch and its associated links are shown in sketch (a). Since the cut set for branch 11 involves links 1, 3 and 4, its insertion into the partial graph of sketch (a) is clear and leads to the result in sketch (b). In this step there are, of course, two points at which branch 11 with its associated link 3 can be inserted, but both of these evidently lead to the same topological result. Incidentally, link 3 like tree-branch 11, is a new addition to the partial graph of sketch (a) and hence appears simply in parallel with that tree branch.

Cut set 10 involves no new links but only 1 and 3 which converge at a common node in the partial graph of sketch (b); hence its insertion is clear and unambiguous and leads to the graph of part (c) in this sequence of sketches. In like manner the insertion of branch 9 picks up links 2 and 3, and introduces link 5 as shown in sketch (d). Next, branch 8 picks up links 3 and 5, and introduces link 6 (sketch e); and finally branch 7 is inserted at the node where links 3, 5 and 6 converge. This step yields the completed graph in sketch (f) which possesses the given cut-set matrix whence the submatrix $\alpha_{n\ell}$ is derived by a process described above.

3. The mechanism for putting the rows of $\alpha_{n\ell}$ into proper order.

The key to this process lies in discovering a test which will reveal those of the cut sets in a given α matrix (in the form 7) that represent branches converging toward single nodes, for these single-node cut sets are the first set of tree tips (Fig. 1). When these are known, we strike out the pertinent rows in α and also all columns corresponding to links associated exclusively with these tips (after deletion of the pertinent rows, these are null columns). It is helpful to recognize in this connection that striking out a row in α corresponds physically to setting the pertinent tree-branch voltage (or node-pair voltage) equal to zero and hence implies the short-circuiting of that tree branch. Striking out a column, on the other hand corresponds to setting the pertinent link current equal to zero and hence implies the opening or removal of that link.

Therefore, when we have found (by a process to be described shortly) the rows in α corresponding to single-node cut sets and then delete these rows together with those columns in which nonzero elements are restricted to (i.e., are unique to) these rows, we are in effect removing the last year's growth (first set of tips) from the tree together with all the links that attach exclusively thereto. The abridged form of α-matrix thus produced, pertains to that portion of the original graph that remains after the tree tips and uniquely attached links are pruned away. With reference to Fig. 1 we may say that the first set of tips shown in part (a) have been removed and there remains the abridged form of the tree in part (b) together with its associated links. This abridged tree again has tips which can be found by treating the corresponding abridged α matrix in the same manner as α is treated in the first step. Thus we discover the second set of tips and a second abridged form for the α matrix. Continuation finally leads to a tree for which all branches are tips.

We are now in a position to reconsider the original α matrix and arrange its rows in groups in which the first represents the first set of tips, the second represents the second set of tips, and so forth. Like the matrix in Fig. 2, this result yields the desired "ordered form" from which the graph may be constructed by inspection.

Except for some minor details, it remains only to describe the process by which we discover those rows in a given α matrix corresponding to branches converging toward single nodes. This process is based upon the observation that if the branches in any other cut set are removed, the graph is divided into two separate parts and the associated node conductance matrix pertains not to a single network but to two electrically independent ones. With proper branch numbering, such a matrix has a form like

$$
G = \begin{bmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 & 0 \\ g_{31} & g_{32} & g_{33} & 0 & 0 \\ 0 & 0 & 0 & g_{44} & g_{45} \\ 0 & 0 & 0 & g_{54} & g_{55} \end{bmatrix} \tag{10}
$$

For any other branch numbering sequence the G matrix will be this one with some row and corresponding column interchanges. For example, interchanging rows and columns 2 and 4, the matrix 10 looks like

$$
G' = \begin{bmatrix} g_{11} & 0 & g_{13} & g_{12} & 0 \\ 0 & g_{44} & 0 & 0 & g_{45} \\ g_{31} & 0 & g_{33} & g_{32} & 0 \\ g_{21} & 0 & g_{23} & g_{22} & 0 \\ 0 & g_{54} & 0 & 0 & g_{55} \end{bmatrix} \tag{11}
$$

Although it is not so apparent here that two electrically independent networks are involved, a simple way of detecting this property will presently be given.

In the meantime we observe that since element values are unimportant in these manipulations we can assume that all branches in the pertinent network are 1-ohm conductances. According to well well known procedure, the G matrix is simply the Grammian formed from the rows of the pertinent α matrix. Hence if we want to know whether a particular row in the given α matrix is or is not a single-node cut set, we delete that row together with all columns in which that row has non-zero elements, and then form the Grammian determinant. If it assumes a form like 10, we know that the particular row is not a single-node cut set. If it assumes a form like 11 and we can't be sure that it derives from a form like 10 merely through row and column interchanges, then we form the Grammian once more. If the pattern of zeros persists then we can conclude that it does derive from a form like 10 through appropriate row and column interchanges because the zero pattern in 10 is invariant to the Grammian process and so is that in 11 because the congruent transformation from 10 to 11 and the Grammian process of formation are interchangeable operations.

Thus suppose T is an elementary transformation matrix formed from the unit matrix by an interchange of the pertinent rows in it. Then

$$
G' = T \times G \times T_t \tag{12}
$$

The Grammian formed from G is $G \times G_t$ and that formed from G' is

$$
G' \times G'_t = T \times G \times T_t \times T \times G_t \times T_t \tag{13}
$$

But, for the elementary transformation matrix T, its transpose is also its inverse; hence

$$
G' \times G'_t = T \times G \times G_t \times T_t \tag{14}
$$

which substantiates the statement that the zero pattern in 11 is invariant to the Grammian process.

In the given α matrix we can thus sort out the rows that correspond to tree tips. For, if such a row (together with the columns containing nonzero elements) is deleted from α and the Grammian is formed, we may, of course, get a G matrix that also displays some zero elements, but upon repeated formation of the Grammian these disappear, while for cut sets other than those pertaining to tree tips the zero pattern persists.

The computational labor involved in carrying out this process can be materially reduced by observing in the formation of Grammian determinants that we do not need to compute values for the elements but merely note whether they are zero or nonzero and record them by an $\underline{x}$ if nonzero, by an $\underline{0}$ if zero. This process is fast and effortless.

Whether an element g_{sk} in the G matrix is nonzero or zero depends in general upon cut sets $\underline{s}$ and $\underline{k}$ having one or more links in common or not having any links in common. Zeros might, of course, in some cases come about through branch conductances having just the right values, but this is not the kind of decoupling condition we are after here, and so we not only can but should ignore element values.

In carrying out the sorting procedure described above, it is clear that one does not need to work with the entire α matrix; the submatrix $\alpha_{n\ell}$ is adequate since only the principal diagonal terms in G are affected by the submatrix u_n in 7, and their values are unimportant so long as they do not become zero, a condition implying that a tree branch is "waiving in the breeze" so to speak with no links attached to it. While such a situation does not normally occur in the given matrix, it can result in a subsequent reduced form obtained after deletion of columns pertinent to the links in a given cut set which is being tested. Here we can, if necessary, supply the missing contribution from the elements in u_n by inspection. In the following the term "cut set" will, therefore, be used to refer simply to the rows in $\alpha_{n\ell}$.

4. Some necessary preliminary manipulations.

A row in $\alpha_{n\ell}$ that contains a single nonzero element may always be interpreted topologically as resulting from a link in series with a tree branch which is a tip. The fact that it need not necessarily be so interpreted is illustrated in Fig. 3 in which tree branches are drawn as solid lines and links are dotted. In part (a) we have a tree tip in series with a link while in (b) we do not; yet both graphs clearly have the same cut-set matrix. We regard (b) as a trivial variant of (a) which can be recognized by inspection and hence need not receive further consideration.

It is, however, important to note that the link involved in this cut set must also be involved in some other cut set and hence must yield a nonzero element in another row of the matrix $\alpha_{n\ell}$. If the test described above is applied to this other row, then the one containing the single nonzero element is reduced to a row of zeros and the method will indicate that the graph is separated into two electrically independent parts whereas the other cut set may actually have single-node character. In this event the test will lead to an error.*

Such errors may be avoided by initially deleting from $\alpha_{n\ell}$ all rows containing single nonzero elements and recognizing that the pertinent tree branches form the first set of tips to be removed from the tree together with those links that attach exclusively thereto. Removal of the latter corresponds to deleting those columns in which nonzero elements are confined to the pertinent rows.

As a further preliminary measure we can also delete from $\alpha_{n\ell}$ all columns containing single nonzero elements, for this element represents a link in parallel with, and hence unique to, the tree branch pertinent to the row in which it occurs. So far as the sorting process is concerned this tree-branch-and-parallel-link combination may be replaced by the tree branch alone and hence nothing is lost through deleting such single-element columns.

*Actually this situation is not serious since in a straight-forward application of the sorting process the tree tips with single links attached will be revealed; and after their rows are deleted, continuation of the sorting procedure then reveals the other tips to which these single links are attached. However, the procedure suggested in the following discussion shortens the work and hence arrives more rapidly at the same end result.

This step may, however, leave the matrix with some single-element rows again. That is to say, the plucking away of links in parallel with some tree branches may leave these with only single links attached. After recognizing them as another set of tree tips and deleting their corresponding rows, we may again find single-element columns in the remaining matrix, etc. When this sort of preamble ceases to be applicable, the sorting process is begun.

A simple example in which the preamble exhausts the α matrix completely is given by the graph and cut-set schedule in Fig. 4. Again tree branches are the solid lines and links are dotted. Here we begin by striking out the first row. Geometrically this act corresponds to shrinking branch 6 until the nodes at its ends coincide (because, as mentioned above, it implies setting the voltage across this tree branch to zero). From the graph in Fig. 4 we see that now link 1 is in parallel with the branch 7; and in the reduced α matrix column 1 contains a single nonzero element. Striking out this column is equivalent geometrically to removing link 1 (since it corresponds to forcing link current 1 to be zero). We now have tree branch 7 in series with link 2, and in the reduced α matrix the row for cut set 7 contains a single nonzero element.

Continuation of this process clearly exhausts the entire α matrix and reveals its ordered form from which the graph is readily regained through successive insertion of tree branches as is done in Fig. 2.

The proper order in which tree branches are inserted is in general the reverse of the order in which the rows are stricken out; however, in this special example the opposite order works just as well.

If a row in the cut-set matrix is duplicated, the pertinent tree branches are is series, and we may tentatively strike out either one since its ultimate reinsertion presents no problem. If a column in the cut-set matrix is duplicated. The pertinent links are in parallel. Here we not only may but must strike out one of these columns, for doing so may again leave the matrix with some single-element rows representing another set of tree tips. Unless the duplicate columns and subsequent single-element rows are stricken out, the sorting process will yield a false conclusion for the reason given above.

Fig. 5 illustrates a simple situation of this sort. Here the tip represented by branch a yields a single-element row in α, and its deletion (corresponding to shrinking branch a) leaves two links in parallel and hence yields two identical columns in α. If the sorting process is at this stage applied to the row corresponding to branch c it will fail to indicate that the pertinent cut set has single-node character, for branch b will thereby be revealed as a separate part. However, if we delete one of the columns corresponding to the two parallel links then the row for branch b becomes a single-element row, and is in turn also deleted, thus recognizing branch b as a secondary tip. Following this step, the sorting procedure yields true results.

The preamble consisting of striking out single-element rows and columns and/or duplicate rows and columns, perhaps in a continuing sequence until no further such deletions are possible, precedes each application of the sorting process. The latter determines those cut sets having single-node character (the next set of tree tips). When these are found, the pertinent rows in α are stricken out together with those columns that then have no nonzero elements left. Thus the preamble followed by one application of the sorting process, yielding a reduced α matrix, together form one cycle in the total reduction method whereby the growth pattern of the tree is revealed. In an extensive situation, this cycle may need to be repeated several times.

In the general process of constructing a graph once the ordered form of the cut-set matrix is established, there will be times when one or more links are inserted in parallel with the pertinent tree branch, namely when the particular cut set contains links not previously introduced (like link 3 in cut set 11 or link 5 in cut set 9 in the example of Fig. 2). If an ambiguity exists as to the point at which the tree branch should be inserted, it is resolved by finding the nearest subsequent cut set containing the same link (or links) and noting those of its remaining links that are already present in the tentative graph. The particular tree-branch-and-parallel-link combination must clearly be inserted at that point where these remaining links converge, since it is otherwise impossible properly to insert the tree branch associated with the subsequent cut set.

If this resolution technique leads to an impasse, then no pertinent graph exists; if it is indecisive, then the ambiguity can be ignored and some trivial variants in the construction of the final graph are possible.

5. An illustrative example

Consider the matrix $\alpha_{n\ell}$ given by the following schedule:

	1	2	3	4	5	6	7	8	9	10
11	-1	0	0	0	0	0	1	0	0	0
12	0	-1	1	0	0	0	0	0	1	0
13	0	0	0	0	-1	0	0	0	0	0
14	-1	0	1	0	0	0	1	-1	1	0
15	-1	0	0	0	0	0	0	0	0	0
16	0	0	0	-1	1	0	0	0	0	0
17	0	0	0	0	0	-1	0	0	0	0
18	1	0	0	0	-1	0	-1	1	0	1
19	-1	0	0	0	0	0	0	0	0	0
20	0	0	0	0	-1	1	0	0	0	1
21	0	-1	0	0	0	0	0	0	0	0
22	0	0	-1	0	1	0	0	0	0	-1
23	0	0	0	-1	0	0	0	0	0	0
24	0	0	0	0	0	0	1	-1	0	0
25	0	0	-1	0	0	0	0	0	0	0
26	-1	1	0	0	0	0	1	0	0	0

(15)

It relates to a 26-branch graph having a total of 17 nodes or 16 tree branches, and 10 links. The latter are numbered consecutively from 1 to 10 and the tree branches from 11 to 26.

Regarding the preamble, we note that rows for tree branches 13, 15, 17, 19, 21, 23 and 25 contain single nonzero elements, and after striking out these rows we find that columns 4 and 6 have single nonzero elements. Striking these out also, leaves the row for tree branch 16 with a single nonzero element and so it is next deleted. At this point columns 5 and 10 are identical, and after striking out 10, the row for tree branch 20 contains a single nonzero element. The deletion of this row completes the preamble and leaves schedule 15 in the form

	1	2	3	5	7	8	9
11	-1	0	0	0	1	0	0
12	0	-1	1	0	0	0	1
14	-1	0	1	0	1	-1	1
18	1	0	0	-1	-1	1	0
22	0	0	-1	1	0	0	0
24	0	0	0	0	1	-1	0
26	-1	1	0	0	1	0	0

(16)

It will be noticed in this example that we are including algebraic signs with the elements. This is done only to allow reference arrows to be assigned to the final graph. In the present manipulations they play no part and should be ignored.

In order to see whether the first row in 16 is a single-node cut set, we remove it and columns 1 and 7. The remainder

	2	3	5	8	9
12	-1	1	0	0	1
14	0	1	0	-1	1
18	0	0	-1	1	0
22	0	-1	1	0	0
24	0	0	0	-1	0
26	1	0	0	0	0

(17)

yields a Grammian matrix of the form

$$\begin{bmatrix} x & x & 0 & x & 0 & x \\ x & x & x & x & x & 0 \\ 0 & x & x & x & x & 0 \\ x & x & x & x & 0 & 0 \\ 0 & x & x & 0 & x & 0 \\ x & 0 & 0 & 0 & 0 & x \end{bmatrix} \tag{18}$$

Forming the Grammian again, we get

$$\begin{bmatrix} x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & 0 \\ x & x & x & x & x & x \\ x & x & x & x & x & 0 \\ x & x & 0 & x & 0 & x \end{bmatrix} \tag{19}$$

from which it is evident that the next one will have no zero elements. Hence the zero pattern in 18 does not indicate that the graph has more than one separate part, and we conclude that branch 11 is a tree tip.

Next we strike out the second row in 16 together with columns 2, 3 and 9, leaving

	1	5	7	8
11	-1	0	1	0
14	-1	0	1	-1
18	1	-1	-1	1
22	0	1	0	0
24	0	0	1	-1
26	-1	0	1	0

(20)

The resulting Grammian matrix has the form

$$\begin{bmatrix} x & x & x & 0 & x & x \\ x & x & x & 0 & x & x \\ x & x & x & x & x & x \\ 0 & 0 & x & x & 0 & 0 \\ x & x & x & 0 & x & x \\ x & x & x & 0 & x & x \end{bmatrix} \tag{21}$$

Upon forming the Grammian once more it is clear that this zero pattern disappears. Hence tree branch 12 is also a tip.

Deleting the third row in 16 together with columns 1, 3, 7, 8 and 9 leaves

	2	5
11	0	0
12	-1	0
18	0	-1
22	0	1
24	0	0
26	1	0

(22)

and yields a Grammian matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & x \\ 0 & 0 & x & x & 0 & 0 \\ 0 & 0 & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & x \end{bmatrix} \qquad (23)$$

This zero pattern clearly is invariant to the Grammian process. Hence tree branch 14 is not a tip.

Continuing in this manner, we find straight-forwardly that branch 18 is not a tip; branches 22 and 24 are tips; and that branch 26 is not a tip.

From schedule 16 we now delete rows for branches 11, 12, 22 and 24, leaving

	1	2	3	5	7	8	9
14	-1	0	1	0	1	-1	1
18	1	0	0	-1	-1	1	0
26	-1	1	0	0	1	0	0

(24)

The fact that no null columns appear here shows that none of the links in the graph pertinent to schedule 16 attach exclusively to the tree tips.

Applying the preamble to 24 we strike out columns for links 3 and 9 which are in parallel with tree branch 14; for link 2 in parallel with the tree branch 26; and for link 5 in parallel with tree branch 18. Schedule 24 then assumes the form

	1	7	8
14	-1	1	-1
18	1	-1	1
26	-1	1	0

(25)

which tells us that at this stage the tree branches 14 and 18 are in series; links 1 and 7 are in parallel and common to all three remaining cut sets, while link 8 is common to and hence in parallel with the series combination of 14 and 18. Tree branches 14, 18 and 26 are obviously in series, with either 14 or 18 in the middle.

We can now put the rows in schedule 15 in ordered form, starting with the single-element rows which we will write in the sequence

$$15,\ 19,\ 21,\ 25,\ 23,\ 13,\ 17 \tag{26}$$

These we can regard as the first set of tips or the outermost twigs of the tree. Next in order come tree branches

$$16,\ 20,\ 11,\ 12,\ 22,\ 24 \tag{27}$$

and finally 18, 26 and 14 in that order, in which 18 and 14 can be interchanged, but we will arbitrarily choose 14 as representing the first year's growth of the tree. The ordered form of schedule 15 is thus revealed to be

	1	2	3	4	5	6	7	8	9	10
15	-1	0	0	0	0	0	0	0	0	0
19	-1	0	0	0	0	0	0	0	0	0
21	0	-1	0	0	0	0	0	0	0	0
25	0	0	-1	0	0	0	0	0	0	0
23	0	0	0	-1	0	0	0	0	0	0
13	0	0	0	0	-1	0	0	0	0	0
17	0	0	0	0	0	-1	0	0	0	0
16	0	0	0	-1	1	0	0	0	0	0
20	0	0	0	0	-1	1	0	0	0	1
11	-1	0	0	0	0	0	1	0	0	0
12	0	-1	1	0	0	0	0	0	1	0
22	0	0	-1	0	1	0	0	0	0	-1
24	0	0	0	0	0	0	1	-1	0	0
18	1	0	0	0	-1	0	-1	1	0	1
26	-1	1	0	0	0	0	1	0	0	0
14	-1	0	1	0	0	0	1	-1	1	0

(28)

Starting with branch 14 and working upward, the graph is now easily constructible. The sketches in Fig. 6, parts (a) through (i), illustrate the successive insertion of tree branches up to the group represented by the first seven rows which are simply in series with links 1 through 6. The completed graph is shown in Fig. 7 in which reference arrows are added in agreement with the algebraic signs in 15 or 28.

A number of trivial variants in this final graph are possible as is obvious from some of the steps in Fig. 6 where branches can be inserted at alternate points. For example, in the step from (g) to (h), branch 20 can alternately be inserted in series with 22; and in the step from (h) to (i), branch 16 can be inserted in series with 20.

6. Concerning the existence of a solution

The fact that this method must yield a graph if one exists at all is self evident and hardly needs formal proof. The existence of a graph is synonomous with assuming that the given α or β matrix was obtained from a graph in the first place. If it was, then the method given here will surely regain that graph.

Thus if we start with a graph, pick a tree, form an α matrix, and then premultiply by a non-singular transformation matrix that produces independent linear combinations of rows; and if we repeat this process for all geometrically distinct tree configurations (which as pointed out above is not a prohibitively large number), then we will form all possible α matrices that can represent

the given graph except for trivial variants resulting from renumbering of the branches. Unless the given α matrix is one of these possible ones, the original graph cannot be regained; and if it is one of these, then the procedure, discussed in art. 1 above, for transforming α into a canonic form must yield that form corresponding to one of the geometrically distinct tree configurations whence α was derived in the first place. Any other conclusion clearly leads to a contradiction.

Putting the given α matrix into a cononic form according to the method described in art. 1 involves the choice of a particular set of n independent columns, and in turn implies the choice of one of the geometrically distinct tree configurations possible for the graph associated with this matrix.* The method outlined in this paper then discovers the growth pattern of that tree and reconstructs it according to this pattern.

7. Concluding remarks.

In farming, there is a saying among old timers to the effect that "all signs fail in dry weather". In network theory we recognize an analogous situation to the effect that for every stated general property or procedure one can always find a sufficiently degenerate example to which the statement does not apply or the procedure is not applicable; in other words, the most carefully contrived method will fail in an extreme situation.

In an intelligent approach to the use of any procedure or method, however, this fact is hardly of real concern since the highly degenerate character of such an exceptional situation can always be used as a means for its detection and resolution.

Regarding the method discussed in this paper, a "culprit" that in this sense defies the operation of our sorting procedure is shown in Fig. 8. This graph consists of three linear trees hinged at the central node, and having no links from one tree to another except the branches numbered 1,2,3 connecting the outermost tips. Except for these three branches we have three independent graphs with the central node in common.

A linear tree by itself has two tips, namely its two ends. In the arrangement of Fig. 8 one end of each tree is at the central node and hence is no longer a tip. The only remaining tips, therefore, are those three where the links 1, 2, 3 converge in pairs. But if we ask our sorting process to find one of these by deleting that row in $\alpha_{n\ell}$ pertinent to, say, tree branch a together with columns representing links converging at the top node, then the subgraph associated with the remainder of the tree containing branch a becomes decoupled from the rest of the graph and the sorting process fails to indicate that the topmost node is a tip. In fact the sorting process fails to reveal any tips at all in this situation.

This fact, however, is precisely the way in which a degeneracy of this sort reveals itself. If we use the letters a, b and c to refer to the three trees in the graph of Fig. 8, then the pertinent matrix $\alpha_{n\ell}$ may (for a specially chosen numbering of the branches) be seen to have a form like the following schedule

* The number of ways in which n independent columns can be chosen is, of course, much larger than the number of geometrically distinct trees associated with the corresponding graph, since each geometrical tree pattern can in general be realized with many different branch combinations. However, in our problem these are trivial variants resulting from renumbering of branches.

	1	2	3	α	β	γ
a	x x x x x		x x x x x	X	0	0
b	x x x x x	x x x x x		0	X	0
c		x x x x x	x x x x x	0	0	X

(29)

in which the letters α, β, γ refer to links associated respectively with trees a, b, c and in these columns the rectangles marked with zeros contain only null elements while those marked with a criss-cross contain the normal distributions of zero and nonzero elements. The distribution of "exes" in the first three columns is consistent with the fact that link 1 is contained in all cut sets pertinent to trees a and b and is absent in all cut sets in tree c; and analogous statements for links 2 and 3. The "exes" and blank spaces in these columns thus indicate consistently nonzero or consistently zero elements in these groups.

In the analogous situation involving only two hinged linear trees (or subgraphs) and a single link joining their extremities, the column for this link immediately reveals itself as that one in which all elements are nonzero. Upon its deletion and formation of the Grammian, separation of the two hinged subgraphs becomes a simple matter.

Where three hinged subgraphs are involved as in Fig. 8, we note from the form of schedule 29 that no matter how much we may interchange rows and columns, we can always recognize its origin in the form of 29 by applying the following reasoning. First we observe that the zeros in columns 1, 2, 3 form mutually exclusive sets and that the total number of zeros in these columns equals n, the total number of rows in the schedule. Comparison of this situation with the zero pattern in the rest of the schedule shows that no columns chosen from the α, β or γ groups can singly or in any combination replace columns in the 1, 2, 3 group since their zeros do not form mutually exclusive sets either amongst themselves or in combination with one or two of the columns 1, 2, 3.

A systematic procedure for recognition is the following. Pick any column and compare its zero pattern with all other columns. If the chosen column happens to be 1, 2 or 3, there will be at least two other columns having nonzero elements wherever the chosen one has zero elements, namely the other two columns in the group 1, 2, 3 and perhaps some in the α or β or γ groups (in a mutually exclusive manner). In this event all three columns 1, 2 and 3 are then isolated by the mutually exclusive property of their zero patterns.

On the other hand, if the chosen column does not happen to be in the 1, 2, 3 group then there can at most be only one other column having nonzero elements wherever the chosen one has zero elements and if there is such a one it must be 1, 2 or 3; whereupon the other two in this group are found as just indicated. In any case, the column having the largest number of nonzero elements where the chosen one has zero elements must be 1, 2 or 3; and thus this group is always identifiable with a moderate effort.

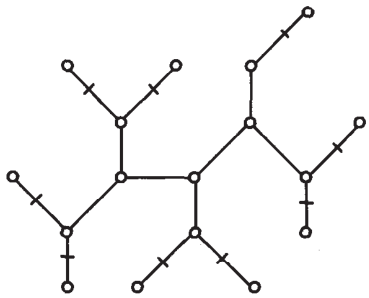
If the given graph consists of more hinged subgraphs and if some of the links like 1, 2 or 3 are missing, a suitably extended version of this same kind of reasoning can evidently be devised to reveal the nature of things and to isolate the columns pertaining to the outermost links. Once these are found, the subgraphs are readily isolated by noting, for example, in schedule 29 that column 1 is orthogonal to all columns in γ and to these alone. The subgraphs are thus identifiable with submatrices in the pertinent $\alpha_{n\ell}$, and their construction is then straight forward.

Another way to "wiggle out of" a difficulty such as this is to change trees by selecting in the given matrix a different set of independent columns in the initial process of converting this matrix to a canonic form, as discussed in art. 1 above. It is not likely that the same kind of highly degenerate situation will occur for a number of different chosen sets of independent columns.

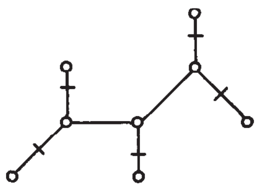
REFERENCES

1. See Introductory Circuit Theory, John Wiley and Sons, 1953, Chapter I.

2. See for example, "Some Generalizations of Linear Network Analysis That Are Useful When Active And/Or Nonbilateral Elements Are Involved," M. I. T., Res. Lab. of Electronics, Quarterly Progress Report of October 15, 1957, pp. 103-114.



(a) GIVEN TREE (3 YEARS OLD)



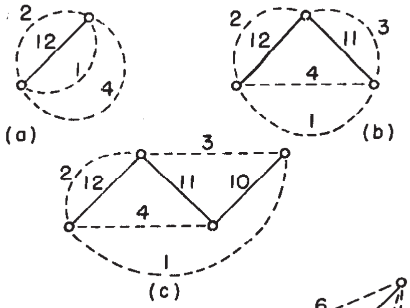
(b) FIRST SET OF TIPS REMOVED
(GROWTH AFTER 2 YEARS)

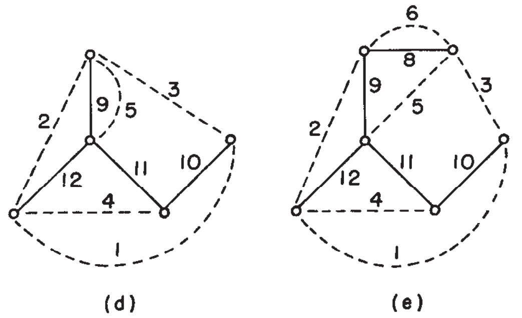
(c) SECOND SET OF TIPS REMOVED
(GROWTH AFTER I YEAR)

(d) CENTRAL NODE OR SEEDLING

Fig. 1

	1	2	3	4	5	6
7	0	0	1	0	1	1
8	0	0	1	0	1	1
9	0	1	1	0	1	0
10	1	0	1	0	0	0
11	1	0	1	1	0	0
12	1	1	0	1	0	0





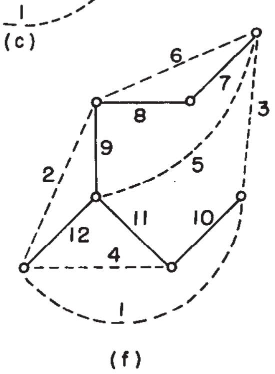
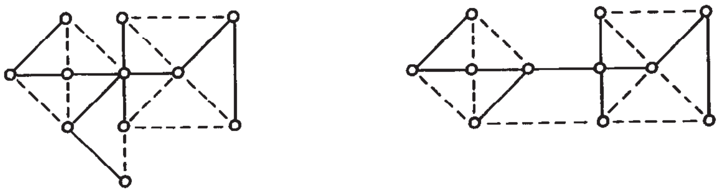


Fig. 2



(a) (b)

Fig. 3

$$a_{n\ell} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array}$$

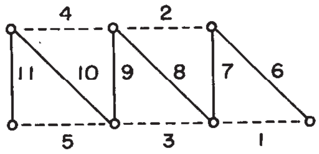


Fig. 4

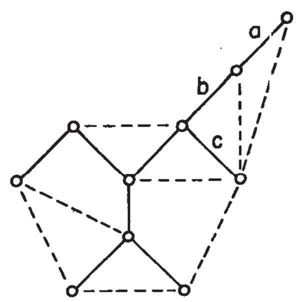


Fig. 5

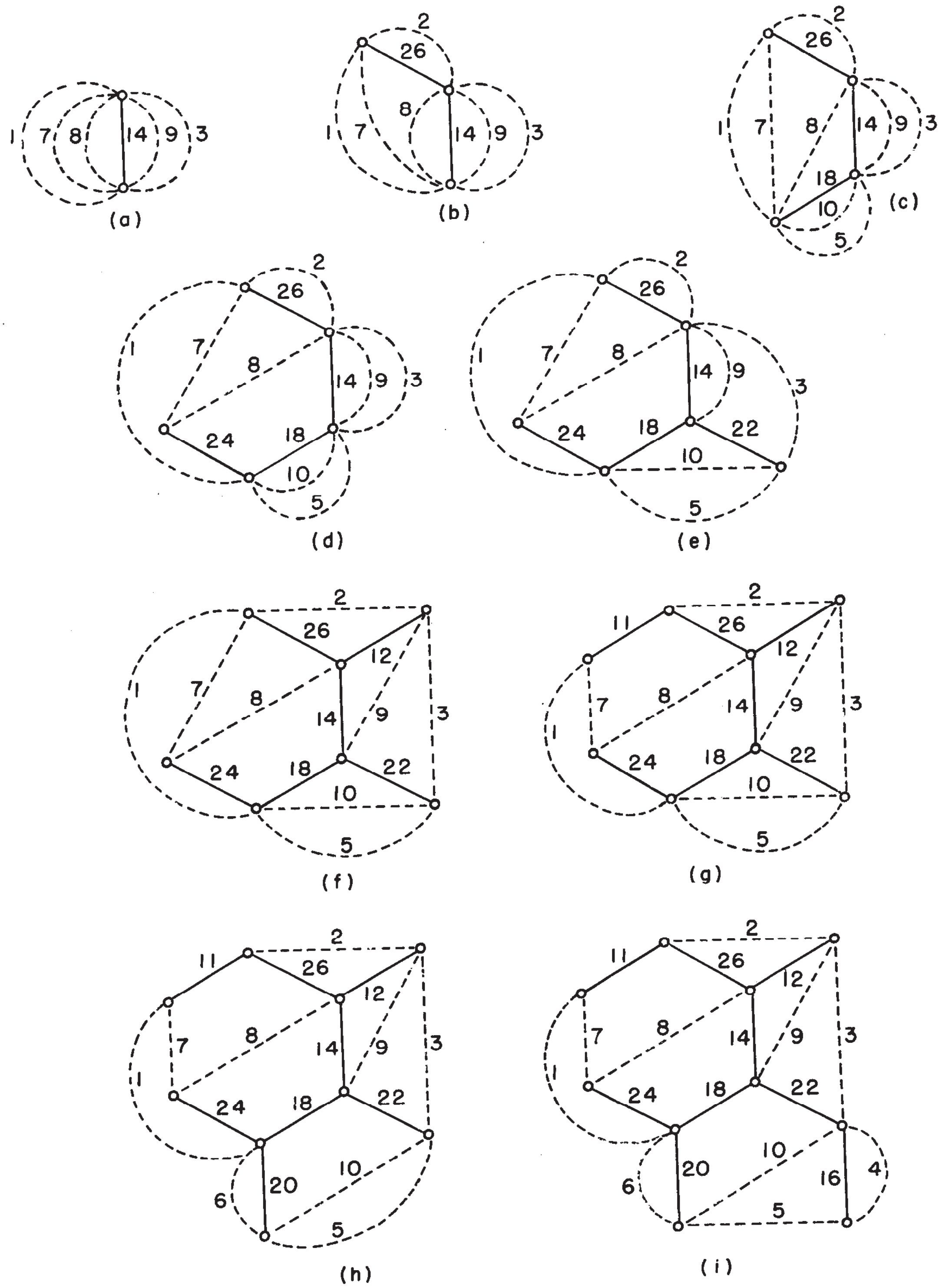


Fig. 6

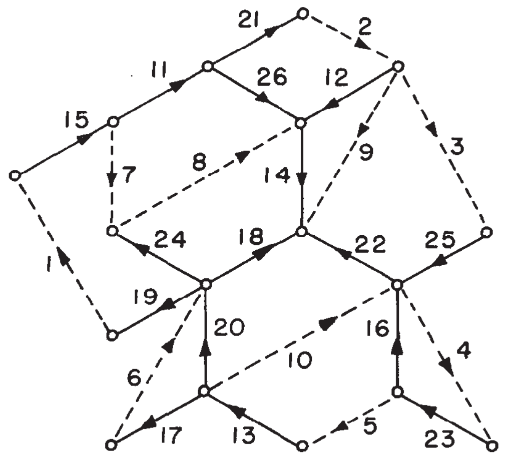


Fig. 7

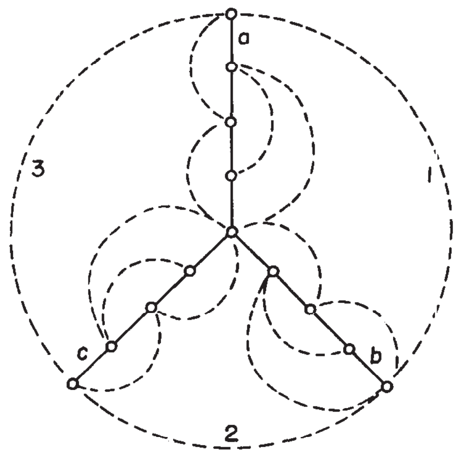


Fig. 8